Fourier Analysis 03-19

Review:

Thm: Let $0<\alpha<1$. Then

$$f_\alpha(x) = \sum_{n=0}^{\infty} 2^{-n\alpha} \cdot e^{i 2^n x}$$

is continuous but nowhere differentiable, and so are the real and imaginary parts of f_α.

- Fourier series of a function on $[a,b]$.

Let $f \in \mathcal{R}[a,b]$. Define

$$\hat{f}(n) = \frac{1}{b-a} \int_a^b f(x)\, e^{-\frac{2\pi i n}{b-a} \cdot x}\, dx$$

and

$$f(x) \sim \sum_{n\in\mathbb{Z}} \hat{f}(n)\, e^{\frac{2\pi i n}{b-a}\cdot x} \quad \text{on } [a,b]$$

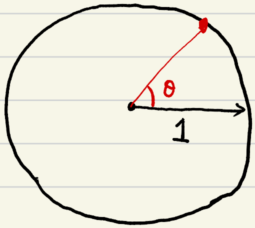
$(b-a)$-periodic function on $\mathbb{R}$

- Parseval identity:

$$\frac{1}{b-a}\int_a^b |f(x)|^2 dx = \sum_{n\in\mathbb{Z}} |\hat{f}(n)|^2.$$

§4.5 Heat equation on the circle.

Model: Heat diffusion on the unit circle.



$-\pi \le \theta \le \pi$

$\theta = 2\pi x$

with $x \in [-\frac{1}{2}, \frac{1}{2}]$

Suppose we know the temperature distribution on the circle at $t=0$.

Question: Find the temperature distribution at time t.

Let $u(x,t)$ be the temperature at x in the time t.

This question was considered by Fourier, in which he applied the idea of Fourier series.

In general, u satisfies the following heat equation

$$\frac{\partial u}{\partial t} = c \cdot \frac{\partial^2 u}{\partial x^2}, \qquad x \in [-\tfrac{1}{2}, \tfrac{1}{2}], \quad t > 0$$

By rescaling t, the above equation can be simplied into

$$\begin{cases} \dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2}, & x \in [-\frac{1}{2}, \frac{1}{2}],\ t > 0. \quad (1) \\ u(x,0) = f(x). & \quad (2) \end{cases}$$

First let us apply the method of "separation of variables".

We try to find out solutions u of (1) with the form $u(x,t) = A(x)\,B(t)$.

Plug $u = A(x)B(t)$ into (1), we obtain

$$A(x)\,B'(t) = A''(x)\,B(t).$$

So

$$\frac{B'(t)}{B(t)} = \frac{A''(x)}{A(x)} = \lambda \quad \text{constant.}$$

First we solve the ODE $A''(x) - \lambda A(x) = 0$ and we would like to find a 1-periodic solution $A(x)$ on $\mathbb{R}$.

Recall

$$A(x) = \begin{cases} c_1 e^{\sqrt{\lambda}x} + c_2 e^{-\sqrt{\lambda}x} & \text{if } \lambda > 0 \\ c_1 e^{i\sqrt{-\lambda}x} + c_2 e^{-i\sqrt{-\lambda}x} & \text{if } \lambda < 0 \\ c_1 x + c_2 & \text{if } \lambda = 0 \end{cases}$$

To ensure that $A(x)$ is 1-periodic, we must have

$$\lambda \leq 0, \quad \text{and} \quad \sqrt{-\lambda} = 2n\pi \quad \text{for} \quad n \in \mathbb{Z}$$

i.e.

$$\lambda = -4n^2\pi^2, \quad n \in \mathbb{Z}.$$

Recall that $B'(t) + 4n^2\pi^2 B(t) = 0$ when $\lambda = -4n^2\pi^2$.

$$B(t) = b\, e^{-4n^2\pi^2 t}.$$

So $$A(x)B(t) = c_1 e^{-4n^2\pi^2 t} \cdot e^{2\pi i n x} + c_2 e^{-4n^2\pi^2 t} e^{-2\pi i n x}, \quad n \in \mathbb{Z}.$$

Next superposing these special solutions, we obtain

$$u(x,t) = \sum_{n \in \mathbb{Z}} a_n e^{-4n^2\pi^2 t} \cdot e^{2\pi i n x}.$$

Suppose u is our solution so that

$$u(x,0) = f(x) \quad \text{where} \quad f \in \mathcal{R}[-\tfrac{1}{2}, \tfrac{1}{2}].$$

We obtain that

$$\sum_{n \in \mathbb{Z}} a_n e^{2\pi i n x} = f(x)$$

Then
$$a_n = \int_{-\frac{1}{2}}^{\frac{1}{2}} f(x)\, e^{-2\pi i n x}\, dx := \hat{f}(n)$$

Hence we obtain a solution

(3) $$u(x,t) = \sum_{n\in\mathbb{Z}} \hat{f}(n)\, e^{-4\pi^2 n^2 t}\, e^{2\pi i n x}, \quad x\in[-\tfrac{1}{2},\tfrac{1}{2}],\ t>0.$$

In what follows, we need to verify that the above series converges, and $u \in C^\infty([-\frac{1}{2},\frac{1}{2}]\times(0,\infty))$.

It is easy to check that the series in defining u in (3) converges absolutely. So we see that $u \in C([-\frac{1}{2},\frac{1}{2}]\times(0,\infty))$

Next we show that $u \in C^\infty$. To prove that, we simply show that $\frac{\partial u}{\partial t}$, $\frac{\partial u}{\partial x}$ exist.

To prove $\frac{\partial u}{\partial t}$ exists, it is enough to see that

(4) $$\sum_{n\in\mathbb{Z}} \frac{\partial}{\partial t}\left(\hat{f}(n)\, e^{-4\pi^2 n^2 t}\, e^{2\pi i n x}\right)$$

converges abs.

i.e $$\sum_{n\in\mathbb{Z}} \hat{f}(n)\cdot\underbrace{(-4\pi^2 n^2)\cdot e^{-4\pi^2 n^2 t}}\cdot e^{2\pi i n x}$$

Hence it converges abs.

(check: $4\pi^2 n^2 t \cdot e^{-4\pi^2 n^2 t}$

$< e^{-nt}$ when n is large enough)

Similarly, $\frac{\partial u}{\partial x}$ exists everywhere and so do the higher order derivatives of u.

Hence $u \in C^\infty([-\frac{1}{2},\frac{1}{2}]\times(0,\infty))$.

$$\frac{\partial u}{\partial t} = \sum_{n\in\mathbb{Z}} (-4n^2\pi^2)\hat{f}(n)\, e^{-4\pi^2 n^2 t} e^{2\pi i n x}$$

$$\frac{\partial^2 u}{\partial x^2} = \sum_{n\in\mathbb{Z}} \hat{f}(n)\, e^{-4\pi^2 n^2 t}\cdot(2\pi i n)^2 e^{2\pi i n x}$$

$$= \sum_{n\in\mathbb{Z}} (-4\pi^2 n^2)\hat{f}(n)\, e^{-4\pi^2 n^2 t} e^{2\pi i n x}$$

So we have $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$, hence ① is satisfied.

In the remaining part we need to verify whether or not u satisfies the initial condition.

Since u is not well-defined on $[-\frac{1}{2}, \frac{1}{2}] \times \{0\}$, the initial condition should be understood as the following:

$$\boxed{\lim_{t \to 0} u(x,t) = f(x).}$$

Prop 1: For $f \in \mathcal{R}[-\frac{1}{2}, \frac{1}{2}]$, we have

$$\lim_{t \to 0} \int_{-\frac{1}{2}}^{\frac{1}{2}} |u(x,t) - f(x)|^2 \, dx = 0.$$

(i.e. $u(x,t)$ converges to $f(x)$ in L^2 sense.

Pf. Notice that by Parseval identity (applying to $u(\cdot,t) - f(\cdot)$)

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} |u(x,t) - f(x)|^2 dx = \sum_{n \in \mathbb{Z}} |\hat{f}(n) e^{-4\pi^2 n^2 t} - \hat{f}(n)|^2$$

(Recall that $u(x,t) = \sum_{n\in\mathbb{Z}} \hat{f}(n) e^{-4\pi^2 n^2 t} e^{2\pi i n x}$)

We need to show that

$$\sum_{n\in\mathbb{Z}} |\hat{f}(n)|^2 \left| e^{-4\pi^2 n^2 t} - 1 \right|^2 \to 0 \quad \text{as } t \to 0.$$

(Justification : $\sum_{n\in\mathbb{Z}} |\hat{f}(n)|^2 < \infty$.

$\forall \varepsilon > 0$, $\exists$ N such that

$$\sum_{|n|>N} |\hat{f}(n)|^2 < \varepsilon.$$

Then we can take a small $\delta > 0$ s.t.

$$\sum_{|n|\leq N} |\hat{f}(n)|^2 \left| e^{-4\pi^2 n^2 t} - 1 \right|^2 < \varepsilon. \text{ for } 0 < t < \delta$$

Hence for $0 < t < \delta$,

$$\sum_{n\in\mathbb{Z}} |\hat{f}(n)|^2 \left| e^{-4\pi^2 n^2 t} - 1 \right|^2$$

$$= \sum_{|n|\leq N} |\hat{f}(n)|^2 \left| e^{-4\pi^2 n^2 t} - 1 \right|^2$$

$$+ \sum_{|n|\geq N} |\hat{f}(n)|^2 \cdot |e^{-4\pi^2 n^2 t} - 1|^2$$

$$\leq \varepsilon + \sum_{|n|\geq N} |\hat{f}(n)|^2 \cdot 1$$

$$\leq 2\varepsilon. \quad \boxtimes$$

<u>Prop 2</u>: Let $f \in \mathcal{R}[-\frac{1}{2}, \frac{1}{2}]$. Assume that f is cts at $x_0 \in [-\frac{1}{2}, \frac{1}{2}]$.

Then $\lim_{t\to 0} u(x_0, t) = f(x_0)$

Moreover if f is cts on the circle, then

$$u(x,t) \rightrightarrows f(x) \quad \text{unif. as } t \to 0.$$

<u>Pf.</u> First notice that

$$u(x,t) = f * H_t(x), \qquad \forall\ t>0,\ x \in [-\tfrac{1}{2}, \tfrac{1}{2}].$$

where $$H_t(x) = \sum_{n\in\mathbb{Z}} e^{-4\pi^2 n^2 t} e^{2\pi i n x}$$

Then we claim that

$\{H_t\}_{t>0}$ is a good kernel on $[-\frac{1}{2}, \frac{1}{2}]$

as $t \to 0$.

Clearly Prop 2 follows from the above claim.

Now we need to check the following.

① $\displaystyle\int_{-\frac{1}{2}}^{\frac{1}{2}} H_t(x)\,dx = 1 \quad \forall\, t>0.$

② $\displaystyle\int_{-\frac{1}{2}}^{\frac{1}{2}} |H_t(x)|\,dx < M \quad \forall\, t>0.$

③ $\forall\, \delta>0,$

$$\int_{\delta<|x|<\frac{1}{2}} |H_t(x)|\,dx \to 0 \quad \text{as } t\to 0.$$

Verification: ①

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} H_t(x)\,dx$$

$$= \int_{-\frac{1}{2}}^{\frac{1}{2}} \sum_{n\in\mathbb{Z}} e^{-4\pi^2 n^2 t} \cdot e^{2\pi i n x}\,dx$$

$$= \sum_{n\in\mathbb{Z}} \int_{-\frac{1}{2}}^{\frac{1}{2}} e^{-4\pi^2 n^2 t}\, e^{2\pi i n x}\,dx$$

$$= 1.$$

② $H_t(x) > 0 \quad \forall\ t>0,\ x \in [-\frac{1}{2}, \frac{1}{2}]$.

This fact comes from the following amazing fact:

$$H_t(x) = \sum_{n\in\mathbb{Z}} \frac{1}{\sqrt{4\pi t}} e^{-\frac{(x+n)^2}{4t}}, \quad \forall\ t>0,\ x\in[-\tfrac{1}{2},\tfrac{1}{2}].$$

③. Let $0<\delta<\frac{1}{2}$.

$$\int_{\delta<|x|<\frac{1}{2}} H_t(x)\,dx$$

$$= \sum_{n\in\mathbb{Z}} \frac{1}{\sqrt{4\pi t}} \int_{\delta<|x|<\frac{1}{2}} e^{-\frac{(x+n)^2}{4t}}\,dx$$

Notice that for any $n\in\mathbb{Z}$, $\delta<|x|<\frac{1}{2}$, one has

$$|x+n| \geq (|n|+1)\,\delta/2$$

Then

$$\frac{1}{\sqrt{4\pi t}} \int_{\delta<|x|<\frac{1}{2}} e^{-\frac{(x+n)^2}{4t}}\,dx$$

$$\leq \frac{1}{\sqrt{4\pi t}} e^{\frac{-(|n|+1)^2\delta^2}{16t}}$$

$$\leq \left(\frac{1}{\sqrt{4\pi t}} \cdot e^{-\frac{\delta^2}{16t}} \right) \cdot e^{-|n| \frac{\delta^2}{16t}}$$

$$\sum_{n \in \mathbb{Z}} \left(\frac{1}{\sqrt{4\pi t}} e^{\frac{-\delta^2}{16t}} \right) \cdot e^{-\frac{|n| \delta^2}{16t}}$$

$$\leq \frac{1}{\sqrt{4\pi t}} \cdot \frac{2 e^{-\delta^2/(16t)}}{1 - e^{-\delta^2/(16t)}} \to 0 \text{ as } t \to 0.$$

$\boxtimes$